ISRAEL JOURNAL OF MATHEMATICS **160** (2007), 371–387
DOI: 10.1007/s11856-007-0067-x

APPROXIMATE VALUES OF THE MULTIPLICITIES IN THE ARM OF THE COCHARACTER SEQUENCE OF $M_{2,1}$

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ABSTRACT

In this paper we study the cocharacter sequence of $M_{2,1}$ and obtain estimates for the multiplicities of the irreducible S_n-characters χ^λ, where λ is any partition with at most 5 parts.

In this paper we investigate the cocharacter sequence of $M_{2,1}$,

$$\chi_n(M_{2,1}) = \sum_{\lambda\in H(5,4;n)} m_\lambda \chi^\lambda.$$

Note that $H(5,4;n)$ denotes the partitions of n in the 5×4 hook, i.e., those in which only the first five parts can be greater than four. For this reason it is reasonable to focus on partitions of height at most five. Let y_λ be the multiplicity of χ^λ in the Young derived sequence of the cocharacter sequence of 2×2 matrices. The coefficient y_λ is a polynomial of degree 7 in the parts of λ, with leading terms computed in equation (3), below. Then we prove that for some constant C,

$$1/6 y_\lambda \le m_\lambda \le C y_\lambda, \tag{1}$$

for all partitions of height between 2 and 5 with at least two parts greater than or equal to 2. The same bounds also hold for the multiplicities in the

* The author's work was supported by both the Faculty Research Council of De Paul University and the National Security Agency, under Grant MDA904-500270. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.
Received September 29, 2005 and in revised form March 14, 2006

trace cocharacter. More generally, given any fixed height h, there exists a constant $C = C(h)$ such that if λ is a partition of height at most h and μ is the first 5 parts of λ, then

$$m_\lambda \leq C y_\mu.$$

For both the upper and lower bound, the techniques we use to prove our main theorem can be used to obtain better bounds.

This work suggests a conjecture for the cocharacters in the arm of any $M_{k,\ell}$. Given any two sequences $\phi = \{\phi_n\}_n$ and $\psi = \{\psi_n\}_n$ of S_n-characters we can define the tensor product $\phi\hat{\otimes}\psi$ via $(\phi\hat{\otimes}\psi)_n = \sum_{i+j=n} \phi_i\hat{\otimes}\psi_j$. We note, for later reference, that this operation is commutative and associative. Let $\chi(F)$ be the cocharacter sequence of the field F, so $\chi_n(F)$ is the irreducible character $\chi^{(n)}$ Then, the Young derived character of ϕ is $\chi(F)\hat{\otimes}\phi$. Keeping in mind that the trace cocharacter sequence and the cocharacter sequence of 2×2 matrices are approximately equal, equation (1) can be thought of as saying that the arm of the cocharacter of $M_{2,1}$ is bounded above and below by constants times $\chi(F)\hat{\otimes}\chi(M_2(F))$. Seen in this light, equation (1) suggests this conjecture.

Conjecture: *Let $\chi^{\mathrm{arm}}(M_{k,\ell})$ be the arm of the cocharacter of $M_{k,\ell}$, namely, the part of the cocharacter corresponding to partitions of height at most $k^2+\ell^2$. Then*

$$\chi^{\mathrm{arm}}(M_{k,\ell}) \approx \chi(M_k(F))\hat{\otimes}\chi(M_\ell(F))$$

in the sense that, for most partitions λ of height at most $k^2+\ell^2$, the multiplicity of χ^λ in $\chi(M_{k,\ell})$ is bounded above and below by constants times its multiplicity in $\chi(M_k(F))\hat{\otimes}\chi(M_\ell(F))$.

1. Preliminaries

1.1 Young Derived Sequences. We recall the definition of Young derived from [7]. Given a sequence of S_n characters $\phi = \{\phi_n\}_{n=0}^\infty$, we construct a new sequence of S_n characters ψ called the Young derived sequence of ϕ and denoted $\psi = \mathcal{Y}(\phi)$. It is defined by

$$\mathcal{Y}(\phi_n) = \sum_{j=0}^{n} \chi^{(j)}\hat{\otimes}\phi_{n-j}.$$

If $\phi_n = \sum_{\lambda\in\mathrm{Par}(n)} m_\lambda\chi^\lambda$, then the multiplicity of χ^λ in $\psi_n = \mathcal{Y}(\phi_n)$ can be computed using Young's rule. It is $\sum m_\mu$ summed over all $\mu \subseteq \lambda$ such that λ/μ

is a horizontal strip. This is equivalent to

$$\sum_{\mu_1=\lambda_2}^{\lambda_1}\sum_{\mu_2=\lambda_3}^{\lambda_2}\cdots\sum_{\mu_n=0}^{\lambda_n} m_\mu .$$

Consider now the case in which each ϕ_n is supported by partitions of height at most k, $\lambda = (\lambda_1, \ldots, \lambda_k)$. This is denoted by $\lambda \in \Lambda_k(n)$. It is useful to extend the notation of Young derivation to functions. Let $m(\lambda)$ be a function on $\bigcup_n \Lambda_k(n)$. Then we can formally define $\mathcal{Y}(m)$ to be a function on partitions of height at most $k+1$ via

$$\mathcal{Y}(m)(\lambda_1, \ldots, \lambda_{k+1}) = \sum_{\mu_1=\lambda_2}^{\lambda_1}\sum_{\mu_2=\lambda_3}^{\lambda_2}\cdots\sum_{\mu_n=0}^{\lambda_n} m(\mu).$$

If we assume that the multiplicities are given by a monomial in the parts of λ, $m(\lambda) = \lambda_1^{n_1}\cdots\lambda_k^{n_k}$, then $\mathcal{Y}(m)(\lambda)$ is given by

$$\sum_{\mu_1=\lambda_2}^{\lambda_1}\mu_1^{n_1}\sum_{\mu_2=\lambda_3}^{\lambda_2}\mu_2^{n_2}\cdots\sum_{\mu_k=\lambda_{k+1}}^{\lambda_k}\mu_k^{n_k}$$
$$= \frac{(\lambda_1^{n_1+1}-\lambda_2^{n_1+1})(\lambda_2^{n_2+1}-\lambda_3^{n_2+1})\cdots(\lambda_k^{n_k+1}-\lambda_{k+1}^{n_k+1})}{(n_1+1)!\cdots(n_k+1)!} + \text{lower order terms.}$$

In order to approximate $\mathcal{Y}(m)$ on polynomial functions, define a map $Y\colon F[\lambda_1, \ldots, \lambda_k] \to F[\lambda_1, \ldots, \lambda_{k+1}]$ given by

$$Y(\lambda_1^{n_1}\cdots\lambda_k^{n_k}) = \frac{(\lambda_1^{n_1+1}-\lambda_2^{n_1+1})(\lambda_2^{n_2+1}-\lambda_3^{n_2+1})\cdots(\lambda_k^{n_k+1}-\lambda_{k+1}^{n_k+1})}{(n_1+1)!\cdots(n_k+1)!},$$

and extend to all of $F[\lambda_1, \ldots, \lambda_k]$ by linearity. Then, for $m(\lambda_1, \ldots, \lambda_k)$ a polynomial,

$$\mathcal{Y}\Big(\sum m(\lambda)\chi^\lambda\Big) = \sum Y(m(\lambda))\chi^\lambda + \text{lower order terms.} \tag{2}$$

In particular, $Y(m)(\lambda) \simeq \mathcal{Y}(m)(\lambda)$ if the differences $\lambda_i - \lambda_{i+1}$ tend to infinity.

The case that is most important for this paper is the trace cocharacter sequence for 2×2 matrices. Here,

$$m_\lambda = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1).$$

The expansion will be a polynomial of degree 3 and $Y(m)$ will be a polynomial of degree 7. The reader may care to check that the leading term in $Y(m_\lambda)$ is

$$1/24(\lambda_4-\lambda_5)(\lambda_3-\lambda_4)(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)$$
$$\times\big(3\lambda_1\lambda_2\lambda_3 - 3\lambda_2\lambda_1\lambda_5 - \lambda_3\lambda_1\lambda_4 - \lambda_1\lambda_3^2 - \lambda_1\lambda_4^2 + 3\lambda_1\lambda_4\lambda_5 - \lambda_2^2\lambda_3 + \lambda_2^2\lambda_5$$
$$+ \lambda_5\lambda_3\lambda_2 + \lambda_3^2\lambda_4 + \lambda_3^2\lambda_5 + \lambda_3\lambda_4^2 - 3\lambda_4\lambda_5\lambda_3 - \lambda_2\lambda_3^2\big).$$

Hence,

$$\begin{aligned} y_\lambda =& 1/24(\lambda_4-\lambda_5)(\lambda_3-\lambda_4)(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)\times \\ & (3\lambda_1\lambda_2\lambda_3 - 3\lambda_2\lambda_1\lambda_5 - \lambda_3\lambda_1\lambda_4 - \lambda_1\lambda_3^2 - \lambda_1\lambda_4^2 + 3\lambda_1\lambda_4\lambda_5 - \lambda_2^2\lambda_3 + \lambda_2^2\lambda_5 + \\ & + \lambda_5\lambda_3\lambda_2 + \lambda_3^2\lambda_4 + \lambda_3^2\lambda_5 + \lambda_3\lambda_4^2 - 3\lambda_4\lambda_5\lambda_3 - \lambda_2\lambda_3^2) + \text{lower order terms.} \end{aligned} \tag{3}$$

1.2 The Inner Product. Given any commutative ring K with 1, we consider the symmetric functions in n variables $x_1,\ldots,x_n$ localized at $x_1\cdots x_n$: $R = K[x_1,\ldots,x_n,(x_1\cdots x_n)^{-1}]^{S_n}$. This ring has a basis consisting of functions of the form $(x_1\cdots x_n)^a S_\lambda(x_1\cdots x_n)$ where S_λ is a Schur function and λ is a partition of height at most $n-1$ and a is an integer. It has an inner product with respect to which this basis is orthonormal. We write the inner product as $\langle f,g\rangle_n$. Formanek showed in [5] how to use this inner product to compute the trace cocharacters for $n\times n$ matrices. Here is his result:

Theorem 1 (Formanek): *Let $m_\lambda^{\mathrm{ptr}}(M_n)$ and $m_\lambda^{\mathrm{mtr}}(M_n)$ be the multiplicities of χ^λ in the pure and mixed trace cocharacter sequences for $n\times n$ matrices over the characteristic zero field, F. Then these multiplicities can be computed as inner products*

$$\begin{aligned} m_\lambda^{\mathrm{mtr}}(M_n) &= \langle S_\lambda(x_i x_j^{-1}), 1\rangle_n \\ m_\lambda^{\mathrm{ptr}}(M_n) &= \left\langle \sum x_i x_j^{-1} S_\lambda(x_i x_j^{-1}), 1\right\rangle_n, \end{aligned}$$

where $i,j=1,\ldots,n$.

In the case of $n=2$, the multiplicities are given by

$$\begin{aligned} m_\lambda^{\mathrm{ptr}}(M_2) &= \langle S_\lambda(x_1x_2^{-1}, x_1^{-1}x_2, 1, 1), 1\rangle_2 \\ m_\lambda^{\mathrm{mtr}}(M_2) &= \langle (2 + x_1x_2^{-1} + x_1^{-1}x_2) S_\lambda(x_1x_2^{-1}, x_1^{-1}x_2, 1, 1), 1\rangle_2. \end{aligned} \tag{4}$$

These multiplicities have been computed explicitly, see [5] and [6]. We will need only the explicit form in the case of the mixed trace and the obvious fact that these are greater than the multiplicities in the pure trace case.

Theorem 2 (Formanek and Procesi): *For all λ of height at most 4,*

$$m_\lambda^{\mathrm{ptr}}(M_2) \le m_\lambda^{\mathrm{mtr}}(M_2) = \omega_1\omega_2\omega_3,$$

where each $\omega_i = \lambda_i - \lambda_{i+1} + 1$.

We conclude this section with a generalization of $\langle\,,\rangle_n$ that we will need in section 2.2. Consider the polynomial ring in two sets of variables $x_1,\ldots,x_n$ and

$y_1, \ldots, y_m$, localized at $x_1 \cdots x_n$ and at $y_1 \cdots y_m$. Let R be the elements of this ring that are symmetric in the x's and y's separately,

$$R = F[x_1, \ldots, x_n, y_1 \cdots y_m, (x_1 \cdots x_n)^{-1}, (y_1 \cdots y_m)^{-1}]^{S_n \times S_m}.$$

This ring has basis

$$(x_1 \cdots x_n)^a (y_1 \cdots y_m)^b S_\lambda(x_1, \ldots, x_n) S_\mu(y_1, \ldots, y_m),$$

where λ is a partition of height at most $n-1$ and μ a partition of height at most $m-1$ and a and b are integers. There is an inner product we denote $\langle\, , \rangle_{n,m}$ with respect to which this basis is orthonormal. This inner product relates to the previous as follows: The algebra R is the tensor product

$$R \cong F[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]^{S_n} \otimes F[y_1, \ldots, y_m, (y_1 \cdots y_m)^{-1}]^{S_m}.$$

Then

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{n,m} = \langle f_1, g_1 \rangle_n \langle f_2, g_2 \rangle_m. \tag{5}$$

2. The Arm of the Cocharacter

2.1 Lower Bound. Let M_2M_1 denoted the algebra of 3×3 matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, where A is a 2×2 matrix and D a 1×1 matrix. The T-ideal of identities of M_2M_1 is the product of the ideals of identities of 2×2 matrices and 1×1 matrices. Moreover, M_2M_1 is p. i. equivalent to a subalgebra of $M_{2,1}$ and so its cocharacter sequence gives a lower bound for that of $M_{2,1}$. That sequence is easy to describe using [3], although we do not know how to compute the trace cocharacters of M_2M_1.

Theorem 3: *The cocharacter of $M_{2,1}$ is greater than or equal to that of M_2M_1 which equals:*

$$\begin{aligned} \chi_n(M_2M_1) =& \chi^{(n)} + \chi_n(M_2(F)) + \chi^{(1)} \hat{\otimes} \sum_{j=0}^{n-1} \chi^{(j)} \hat{\otimes} \chi_{n-1-j}(M_2(F)) \\ &- \sum_{j=0}^{n} \chi^{(j)} \hat{\otimes} \chi_{n-j}(M_2(F)) \end{aligned}$$

This formula can be restated in the more compact form

$$\begin{aligned} \chi(M_{2,1}) &\geq \chi(F) + \chi(M_2(F)) + (\chi^{(1)} - 1) \hat{\otimes} \mathcal{Y}(\chi(M_2(F))) \\ &= \chi(F) + \chi(M_2(F)) + \mathcal{Y}((\chi^{(1)} - 1) \hat{\otimes} \chi(M_2(F))). \end{aligned} \tag{6}$$

Of course, the left hand side of this inequality is supported in the strip $H(5,0)$, which is the arm of the cocharacter.

The cocharacter sequence $\{\chi_n(M_2(F))\}$ was computed by Drensky and Formanek, see [4] and [5]. We record the results.

Theorem 4 (Drensky and Formanek): *The multiplicities in the cocharacter sequence of* 2×2 *matrices are given by*

$$m_\lambda(M_2(F)) = \begin{cases} 1 & \text{if } \lambda=(n)\\ (\lambda_1-\lambda_2+1)\lambda_2 & \text{if } \lambda=(\lambda_1,\lambda_2), \lambda_2\ge 1\\ \lambda_1(2-\lambda_4)-1 & \text{if } \lambda=(\lambda_1,1,1,\lambda_4)\\ (\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)(\lambda_3-\lambda_4+1) & \text{otherwise.}\end{cases}$$

This is the Young derived sequence of $1+\sum\chi^\lambda$*, where* λ *runs over all* $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ *with* $\lambda_2\ge 1$ *and* $\lambda\ne(1,1,1)$

Combining the last part of the Drensky–Formanek theorem with equation (6) yields

$$\chi(M_{2,1}) \ge \mathcal{Y}\Big(2+\sum\{\chi^\lambda|\lambda=(\lambda_1,\lambda_2,\lambda_3)\ne(1,1,1),\lambda_2\ge 1\} + (\chi^{(1)}-1)\hat{\otimes}\chi(M_2(F))\Big). \tag{7}$$

We abbreviate the right hand side as $\mathcal{Y}(\sum n_\lambda\chi^\lambda)$ and we let m'_λ be the function which is 1 on all partitions of height 2 or 3, except for $(1,1,1)$, equal to 2 on the trivial partition and which is zero otherwise.

Lemma 5: *Let* n_λ *be defined as above. If* $\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,1)$ *then* $n_\lambda=m_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}(M_2(F))$*; if* λ *is any other partition of height greater than 4,* $n_\lambda=0$*; and if* λ *has height less than or equal to 4, then* n_λ *is given as in figure 1, where* $\omega_i=\lambda_i-\lambda_{i+1}+1$.

Proof: Given a partition λ, let λ^- be the set of partitions gotten from λ by subtracting 1 from one of the parts. Then

$$n_\lambda = m'_\lambda - m_\lambda + \sum_{\mu\in\lambda^-} m_\mu,$$

where $m=m(M_2(F))$ is as in Drensky–Formanek. If $\lambda_5\ge 2$ or $\lambda_6\ge 1$, then λ^- will contain only partitions of height at least 5 and so $n_\lambda=0$. If $\lambda_5=1$ and $\lambda_6=0$ the only partition of height less than 5 in λ^- is the one received by replacing λ_5 by zero and so $n_\lambda=m_{(\lambda_1,\ldots,\lambda_4)}$.

λ	n_λ
$\lambda_3 \geq 2,\ \lambda_4 > 0$	$3\omega_1\omega_2\omega_3 - \omega_1 - \omega_3$
$\lambda_3 \geq 2,\ \lambda_4 = 0$	$2\omega_1\omega_2\omega_3 - \omega_1 - \omega_3 - \omega_1\omega_2 + 1$
$(\lambda_1, \lambda_2, 1, 1),\ \lambda_2 \geq 3$	$3\omega_1\omega_2\omega_3 - \omega_1 - \omega_3$, where $\omega_3 = 1$
$(\lambda_1, \lambda_2, 1),\ \lambda_2 \geq 3$	$\omega_1(3\lambda_2 - 2) - 1$
$(\lambda_1, 2, 1)$	$4\lambda_1 - 6$
$(\lambda_1, 2, 1, 1)$	$5\lambda_1 - 7$
$(\lambda_1, 1, 1, 1),\ \lambda_1 \geq 2$	$2\lambda_1 - 2$
(1^4)	1
$(\lambda_1, 1, 1)$	$\lambda_1 - 1$
$(\lambda_1, \lambda_2),\ \lambda_2 \geq 2$	$\omega_1(\lambda_2 - 1)$
$(\lambda_1, 1)$	1
$(\lambda_1),\ \lambda_1 \geq 1$	0
(0)	1

Figure 1. Values of n_λ

If $\lambda_3 \geq 2$ and $\lambda_4 \geq 1$ then $m'_\lambda = 0$ and

$$\begin{aligned} n_\lambda =& - m(\lambda_1, \lambda_2, \lambda_3, \lambda_4) + m(\lambda_1 - 1, \lambda_2, \lambda_3, \lambda_4) + m(\lambda_1, \lambda_2 - 1, \lambda_3, \lambda_4) \\ & + m(\lambda_1, \lambda_2, \lambda_3 - 1, \lambda_4) + m(\lambda_1, \lambda_2, \lambda_3, \lambda_4 - 1) \\ =& - \omega_1\omega_2\omega_3 + (\omega_1 - 1)\omega_2\omega_3 + (\omega_1 + 1)(\omega_2 - 1)\omega_3 \\ & + \omega_1(\omega_2 + 1)(\omega_3 - 1) + \omega_1\omega_2(\omega_3 + 1) \\ =& 3\omega_1\omega_2\omega_3 - \omega_3 - \omega_1 \end{aligned}$$

If $\lambda_3 \geq 2$ and $\lambda_4 = 0$, the computation will be the same, except there will be no $m(\lambda_1, \lambda_2, \lambda_3, \lambda_4 - 1)$ term and there will be a $+1$ for the m'_λ. Hence, $n_\lambda = 2\omega_1\omega_2\omega_3 - \omega_3 - \omega_1 - \omega_1\omega_2 + 1$.

We now consider the cases in which $\lambda_3 = 1$. First we consider the cases in which $\lambda_2 \geq 3$. If $\lambda_4 = 1$ the computation is the same as the $\lambda_3 \geq 2$, $\lambda_4 \geq 1$ case with $\omega_3 = 1$ and $m_\lambda = 3\omega_1\omega_2 - 1 - \omega_1$. If $\lambda_2 \geq 3$, $\lambda_3 = 1$ and $\lambda_4 = 0$, then

$$\begin{aligned} n(\lambda_1, \lambda_2, 1) =& m'(\lambda) - m(\lambda_1, \lambda_2, 1) + m(\lambda_1 - 1, \lambda_2, 1) + m(\lambda_1, \lambda_2 - 1, 1) \\ & + m(\lambda_1, \lambda_2) \\ =& 1 - \omega_1(\lambda_2)(2) + (\omega_1 - 1)(\lambda_2)(2) + (\omega_1 + 1)(\lambda_2 - 1)(2) + \omega_1(\lambda_2) \\ =& 3\omega_1\lambda_2 - 2\omega_1 - 1 \end{aligned}$$

If $\lambda = (\lambda_1, 2, 1, 1)$, then $m'_\lambda = 0$ and

$$\begin{aligned} n(\lambda_1, 2, 1, 1) &= -m(\lambda_1, 2, 1, 1) + m(\lambda_1 - 1, 2, 1, 1) + m(\lambda_1, 1, 1, 1) + m(\lambda_1, 2, 1) \\ &= -(\lambda_1 - 1)2 + (\lambda_1 - 2)2 + (\lambda_1 - 1) + (\lambda_1 - 1)4 \\ &= 5\lambda_1 - 7. \end{aligned}$$

If $\lambda = (\lambda_1, 2, 1)$, then $m'_\lambda = 1$ and

$$\begin{aligned} n(\lambda_1, 2, 1) &= 1 - m(\lambda_1, 2, 1) + m(\lambda_1 - 1, 2, 1) + m(\lambda_1, 1, 1) + m(\lambda_1, 2) \\ &= 1 - (\lambda_1 - 1)4 + (\lambda_1 - 2)4 + (2\lambda_1 - 1) + (\lambda_1 - 1)2 \\ &= 4\lambda_1 - 6. \end{aligned}$$

If $\lambda = (\lambda_1, 1, 1, 1)$ with $\lambda_1 \geq 2$, then $m'_\lambda = 0$ and

$$\begin{aligned} n(\lambda_1, 1, 1, 1) &= -m(\lambda_1, 1, 1, 1) + m(\lambda_1 - 1, 1, 1, 1) + m(\lambda_1, 1, 1) \\ &= -(\lambda_1 - 1) + (\lambda_1 - 2) + (2\lambda_1 - 1) = 2\lambda_1 - 2. \end{aligned}$$

And, if $\lambda = (1^4)$, $n((1^4)) = -m((1^4)) + m((1^3)) = 1$.

If $\lambda = (\lambda_1, 1, 1)$ with $\lambda_1 \geq 2$, then $m'_\lambda = 1$ and

$$\begin{aligned} n(\lambda_1, 1, 1) &= 1 - m(\lambda_1, 1, 1) + m(\lambda_1 - 1, 1, 1) + m(\lambda_1, 1) \\ &= 1 - (2\lambda_1 - 1) + (2\lambda_1 - 3) + (\lambda_1) \\ &= \lambda_1 - 1. \end{aligned}$$

If $\lambda = (1, 1, 1)$, then $m'_\lambda = 0$ and $n(1, 1, 1) = -m(1, 1, 1) + m(1, 1) = 0$. We note that $n(\lambda)$ is also equal to $\lambda_1 - 1$ in this case.

We now turn to the height two case. If $\lambda_2 \geq 2$, then $m'_\lambda = 1$ and

$$\begin{aligned} n(\lambda_1, \lambda_2) &= 1 - m(\lambda_1, \lambda_2) + m(\lambda_1 - 1, \lambda_2) + m(\lambda_1, \lambda_2 - 1) \\ &= 1 - \omega_1\lambda_2 + (\omega_1 - 1)\lambda_2 + (\omega_1 + 1)(\lambda_2 - 1) \\ &= \omega_1\lambda_2 - \omega_1 \\ &= \omega_1(\lambda_2 - 1). \end{aligned}$$

If $\lambda = (\lambda_1, 1)$ then

$$\begin{aligned} n(\lambda_1, 1) &= 1 - m(\lambda_1, 1) + m(\lambda_1 - 1, 1) + m(\lambda_1) \\ &= 1 - \lambda_1 + (\lambda_1 - 1) + 1 = 1 \end{aligned}$$

Finally, if $\lambda = (\lambda_1)$, then $m'_\lambda = 0$ and $m(\lambda_1) = m(\lambda_1 - 1) = 1$ and so $n(\lambda) = 0$; unless $\lambda_1 = 0$ in which case, $m'_\lambda = 2$ and there is no $m(\lambda_1 - 1)$ term. The net effect is $n(\lambda) = 1$. ∎

COROLLARY 6: *$6n_\lambda \geq \bar{m}_\lambda$ for all λ except for $\lambda = (1^3)$, $(\lambda_1, 1)$ or (λ_1), $\lambda_1 \geq 1$, where $\bar{m}_\lambda$ is $m_\lambda^{\mathrm{mtr}}(M_2)$ as in Theorem 2.*

Proof: If λ is a partition of height greater than 4, $\bar{m}_\lambda = 0$ and the inequality is automatic. If λ is any partition of height at most 4, then $\bar{m}_\lambda = \omega_1\omega_2\omega_3$ and each $\omega_i \geq 1$. We now consider separately each of the cases of the previous lemma.

In the case of $\lambda_3 \geq 2$ and $\lambda_4 > 0$, or $(\lambda_1, \lambda_2, 1, 1)$ and $\lambda_2 \geq 4$, $6n_\lambda - \bar{m}_\lambda$ equals

$$17\omega_1\omega_2\omega_3 - 6\omega_1 - 6\omega_3 = \omega_1\omega_3(17\omega_2 - 6/\omega_3 - 6/\omega_1) \geq \omega_1\omega_3(17 - 6 - 6) \geq 0.$$

In the case of $\lambda_3 \geq 2$ and $\lambda_4 = 0$, note that $\omega_3 \geq 3$ and so $6n_\lambda - \bar{m}_\lambda$ equals

$$\begin{aligned} 11\omega_1\omega_2\omega_3 - 6\omega_1 - 6\omega_3 - 6\omega_1\omega_2 + 6 &= \omega_1\omega_2(5\omega_3 - 6/\omega_2 - 6) + \omega_3(6\omega_1\omega_2 - 6) \\ &\geq \omega_1\omega_2(15 - 6 - 6) + \omega_3(6 - 6) > 0 \end{aligned}$$

In the case of $\lambda = (\lambda_1, \lambda_2, 1)$ with $\lambda_2 \geq 3$, $\bar{m}_\lambda = 2\omega_1\lambda_2$ and so $6n_\lambda - \bar{m}_\lambda$ equals

$$16\omega_1\lambda_2 - 12\omega_1 - 6 \geq 48\omega_1 - 12\omega_1 - 6 > 0.$$

In the case of $\lambda = (\lambda_1, 2, 1)$, $\bar{m}_\lambda = 4\lambda_1 - 4$ and so $6n_\lambda - \bar{m}_\lambda = 20\lambda_1 - 36$ which is greater than zero since λ_1 must be at least 2.

In the case of $\lambda = (\lambda_1, 2, 1, 1)$, $\bar{m}_\lambda = 2\lambda_1 - 2$ and so $6n_\lambda - \bar{m}_\lambda = 28\lambda_1 - 38$ which again is greater than zero since λ_1 must be at least 2.

In the case of $\lambda = (\lambda_1, 1, 1, 1)$ with $\lambda_1 \geq 2$, $\bar{m}_\lambda = \lambda_1$ and so $6n_\lambda - \bar{m}_\lambda = 11\lambda_1 - 12 > 0$. And, if $\lambda_1 = 1$, then $6n_\lambda - \bar{m}_\lambda = 5$.

If $\lambda = (\lambda_1, 1, 1)$ then $\bar{m}_\lambda = 2\lambda_1$ and so $6n_\lambda - \bar{m}_\lambda = 4\lambda_1 - 6$. This is greater than or equal to zero, unless $\lambda_1 = 1$.

If $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 \geq 2$ then $\bar{m}_\lambda = \omega_1(\lambda_2 + 1)$ and so $6n_\lambda - \bar{m}_\lambda = \omega_1(5\lambda_2 - 7) > 0$. ∎

With the help of this corollary and the previous lemma we can now prove the lower bound on for $m_\lambda(M_{2,1})$ with $\lambda \in H(5, 0)$ which we stated in the introduction.

THEOREM 7: *If $\lambda \in \Lambda_5(n)$ is not of the form (λ_1) or $(\lambda_1, 1)$, then the multiplicity of χ^λ in $\chi_n(M_{2,1})$ is greater than or equal to $\frac{1}{6}y_\lambda$.*

Proof: Comparing the n_λ with $\bar{m}_\lambda = \omega_1\omega_2\omega_3$, the previous corollary states that $n_\lambda \geq \frac{1}{6}\bar{m}_\lambda$ for all λ except $\lambda = (\lambda_1)$, $(\lambda_1, 1)$, or $(1, 1, 1)$. These occur in the computation of $\mathcal{Y}(\bar{m})_\lambda$ and of $\mathcal{Y}(n)_\lambda$ for those λ with λ/μ a horizontal strip, where μ is one of the above. Hence, the possibly problematic λ are those of the form (λ_1, λ_2), $(\lambda_1, \lambda_2, 1)$, and $(\lambda_1, 1^3)$. For all other λ it is immediate that $\mathcal{Y}(n)_\lambda \geq \frac{1}{6}\mathcal{Y}(\bar{m})_\lambda$. We now consider the remaining cases.

Case of $\lambda = (\lambda_1, 1^3)$:

$$\mathcal{Y}(\bar{m})_\lambda = \sum_{i=1}^{\lambda_1}(\bar{m}(i,1^3) + \bar{m}(i,1^2)) = \sum_{i=1}^{\lambda_1}(i+2i) = \frac{3}{2}\lambda_1(\lambda_1+1).$$

On the other hand,

$$\begin{aligned}\mathcal{Y}(n)_\lambda &= \sum_{i=1}^{\lambda_1}(n_\lambda(i,1^3) + n_\lambda(i,1^2)) = \sum_{i=1}^{\lambda_1}(2i-2+i-1)+1 \text{ (for } \lambda=(1,1,1,1))\\ &= \frac{3}{2}\lambda_1(\lambda_1-1)+1.\end{aligned}$$

Hence, $6\mathcal{Y}(n)_\lambda - \mathcal{Y}(\bar{m})_\lambda = \frac{3}{2}\lambda_1(5\lambda_1-7)+6$, which is positive for all $\lambda_1 \geq 1$.

Case of $\lambda = (\lambda_1, 1^2)$:

$$\mathcal{Y}(\bar{m})_\lambda = \sum_{i=1}^{\lambda_1}(\bar{m}(i,1^2) + \bar{m}(i,1^1)) = \sum_{i=1}^{\lambda_1}(2i+2i) = 2\lambda_1(\lambda_1+1)$$

and

$$\mathcal{Y}(n)_\lambda = \sum_{i=1}^{\lambda_1}(n_\lambda(i,1^2) + n_\lambda(i,1)) = \sum_{i=1}^{\lambda_1}(i-1+1) = \frac{1}{2}\lambda_1(\lambda_1+1).$$

Hence, $6\mathcal{Y}(n)_\lambda - \mathcal{Y}(\bar{m})_\lambda = \lambda_1(\lambda_1+1) > 0$.

Case of $\lambda = (\lambda_1, \lambda_2)$: This and the next case are more difficult.

$$\mathcal{Y}(\bar{m})_\lambda = \sum_{\mu_2=0}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1}(\mu_1-\mu_2+1)(\mu_2+1).$$

Making the substitution $\omega_1 = \mu_1 - \mu_2 + 1$ yields

$$\begin{aligned}\mathcal{Y}(\bar{m})_\lambda &= \sum_{\mu_2=0}^{\lambda_2}\sum_{\omega_1=\lambda_2-\mu_2+1}^{\lambda_1-\mu_2+1}\omega_1(\mu_2+1) = \sum_{\mu_2=0}^{\lambda_2}(\mu_2+1)\sum_{\omega_1=\lambda_2-\mu_2+1}^{\lambda_1-\mu_2+1}\omega_1\\ &= \sum_{\mu_2=0}^{\lambda_2}\frac{1}{2}(\mu_2+1)(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2-2\mu_2+2).\end{aligned}$$

In order to do the corresponding computation for n_λ, we assume that $\lambda_2 \geq 2$ and note from table 1 that n_μ is zero if $\mu_2 = 0$ and has different descriptions depending on whether $\mu_2 = 1$ or $\mu_2 \geq 2$. As above, we make the substitution $\omega_1 = \mu_1 - \mu_2 + 1$,

$$\mathcal{Y}(n)_\lambda = \sum_{\mu_2=0}^{\lambda_2} \sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1, \mu_2) = \sum_{\mu_2=1} \sum_{\mu_1=\lambda_2}^{\lambda_1} 1 + \sum_{\mu_2=2}^{\lambda_2} \sum_{\omega_1=\lambda_2-\mu_2+1}^{\lambda_1-\mu_2+1} \omega_1(\mu_2 - 1)$$

$$= \lambda_1 - \lambda_2 + 1 + \sum_{\mu_2=2}^{\lambda_2} \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 - 2\mu_2 + 2)(\mu_2 - 1).$$

We now compute $6\mathcal{Y}(n)_\lambda - \mathcal{Y}(\bar{m}_\lambda)$ equals $\frac{1}{2}(\lambda_1 - \lambda_2 + 1)$ times

$$\sum_{\mu_2=2}^{\lambda_2} (\lambda_1 + \lambda_2 - 2\mu_2 + 2)(5\mu_2 - 7) + 12 - \sum_{\mu_2=0}^{1} (\mu_2 + 1)(\lambda_1 + \lambda_2 - 2\mu 2 + 2)$$

$$= 12 + \sum_{\mu_2=2}^{\lambda_2} (\lambda_1 + \lambda_2 - 2\mu_2 + 2)(5\mu_2 - 7) - (3\lambda_1 + 3\lambda_2 + 2).$$

In order to get an upper bound we replace the sum by its $\mu_2 = 2$ term: $6\mathcal{Y}(n)_\lambda - \mathcal{Y}(\bar{m})_\lambda$ is greater than or equal to $\frac{1}{2}(\lambda_1 - \lambda_2 + 1)$ times

$$3(\lambda_1 + \lambda_2 - 2) + 12 - (3\lambda_1 + 3\lambda_2 + 2) = 4.$$

Case of $\lambda = (\lambda_1, \lambda_2, 1)$ with $\lambda_2 \geq 2$:

$$\mathcal{Y}(\bar{m})_\lambda = \sum_{\mu_2=1}^{\lambda_2} \sum_{\mu_1=\lambda_2}^{\lambda_1} (\bar{m}(\mu_1, \mu_2, 1)) + \bar{m}(\mu_1, \mu_2)$$

$$= \sum_{\mu_2=1}^{\lambda_2} \sum_{\omega_1=\lambda_2-\mu_2+1}^{\omega_1=\lambda_1-\mu_2+1} (2\mu_2\omega_1 + (\mu_2 + 1)\omega_1)$$

$$= \sum_{\mu_2=1}^{\lambda_2} (3\mu_2 + 1) \sum_{\omega_1=\lambda_2-\mu_2+1}^{\omega_1=\lambda_1-\mu_2+1} \omega_1$$

$$= \sum_{\mu_2=1}^{\lambda_2} \frac{1}{2}(3\mu_2 + 1)(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 - 2\mu_2 + 2). \tag{8}$$

In the computation of $\mathcal{Y}(n)_\lambda$, we will need to consider separately $\mu \subseteq \lambda$ of the form $(\mu_1, 2, 1)$, $(\mu_1, 1, 1)$ and $(\mu_1, 1)$. Then,

$$\mathcal{Y}(n)_\lambda = \sum_{\mu_2=1}^{\lambda_2} \sum_{\mu_1=\lambda_2}^{\lambda_1} (n(\mu_1, \mu_2, 1) + n(\mu_1, \mu_2)).$$

Now,

$$\begin{aligned}
&\sum_{\mu_2=1}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,\mu_2,1)\\
&\qquad=\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,1,1)+\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,2,1)+\sum_{\mu_2=3}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,\mu_2,1)\\
&\qquad=\sum_{\mu_1=\lambda_2}^{\lambda_1}\mu_1-1+\sum_{\mu_1=\lambda_2}^{\lambda_1}4\mu_1-6\\
&\qquad\quad+\sum_{\mu_2=3}^{\lambda_2}\sum_{\omega_1=\lambda_2-\mu_2+1}^{\lambda_1-\mu_2+1}(\omega_1(3\mu_2-2)-1)\\
&\qquad=\frac{1}{2}(\lambda_1-\lambda_2+1)(5\lambda_1+5\lambda_2-14)\\
&\qquad\quad+\left(\sum_{\mu_2=3}^{\lambda_2}(3\mu_2-2)\frac{1}{2}(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2-2\mu_2+2)\right)\\
&\qquad\quad-(\lambda_1-\lambda_2+1)(\lambda_2-2)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\mu_2=1}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,\mu_2)&=\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,1)+\sum_{\mu_2=2}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1} n(\mu_1,\mu_2)\\
&=\sum_{\mu_1=\lambda_2}^{\lambda_1}1+\sum_{\mu_2=2}^{\lambda_2}\sum_{\omega_1=\lambda_2-\mu_2+1}^{\lambda_1-\mu_2+1}\omega_1(\mu_2-1)\\
&=(\lambda_1-\lambda_2+1)\\
&\quad+\sum_{\mu_2=2}^{\lambda_2}(\mu_2-1)\frac{1}{2}(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2-2\mu_2+2).
\end{aligned}$$

Hence, $\mathcal{Y}(n)_\lambda$ equals $(\lambda_1-\lambda_2+1)$ times

$$\begin{aligned}
&\frac{1}{2}(5\lambda_1+5\lambda_2-14)+\sum_{\mu_2=3}^{\lambda_2}(3\mu_2-2)\frac{1}{2}(\lambda_1+\lambda_2-2\mu_2+2)-(\lambda_2-2)+1\\
&\quad+\frac{1}{2}(\lambda_1+\lambda_2-4)+\sum_{\mu_2=3}^{\lambda_2}(\mu_2-1)\frac{1}{2}(\lambda_1+\lambda_2-2\mu_2+2)
\end{aligned}$$

$$=3\lambda_1+2\lambda_2-6+\frac{1}{2}\sum_{\mu_2=3}^{\lambda_2}(4\mu_2-3)(\lambda_1+\lambda_2-2\mu_2+2). \tag{9}$$

Combining this equation with equation (8) we compute $6\mathcal{Y}(n)_\lambda - \mathcal{Y}(\bar{m})_\lambda$, by pulling out initial terms in the computation of $\bar{m}_\lambda$ and get $(\lambda_1 - \lambda_2 + 1)$ times

$$6\bigg(3\lambda_1 + 2\lambda_2 - 6 + \frac{1}{2}\sum_{\mu_2=3}^{\lambda_2}(4\mu_2 - 3)(\lambda_1 + \lambda_2 - 2\mu_2 + 2)\bigg)$$
$$-\frac{1}{2}\bigg(\sum_{\mu_2=3}^{\lambda_2}(3\mu_2 + 1)(\lambda_1 + \lambda_2 - 2\mu_2 + 2) + 4(\lambda_1 + \lambda_2) + 7(\lambda_1 + \lambda_2 - 2)\bigg)$$
$$= \frac{1}{2}(25\lambda_1 + 13\lambda_2 - 58) + \frac{1}{2}\sum_{\mu_2=3}^{\lambda_2}(13\mu_2 - 13)(\lambda_1 + \lambda_2 - 2\mu_2 + 2)$$

Each term in the summation is positive and since $\lambda_1 \geq \lambda_2 \geq 2$ the first term is greater than or equal to $50 + 26 - 58$ and so is greater than 0. ∎

Remark 8: There are two remaining cases, $\lambda = (\lambda_1)$ and $\lambda = (\lambda_1, 1)$. Note that $\mathcal{Y}(n)(\lambda_1) = 1$ and $\mathcal{Y}(n)(\lambda_1, 1) = \lambda_1$, and the multiplicity d_λ of the irreducible S_n-module corresponding to λ is also 1 and λ_1 in the respective cases. Since these are equal, $m_\lambda(M_{2,1}) = 1$ for $\lambda = (\lambda_1)$ and $m_\lambda(M_{2,1}) = \lambda_1$ for $\lambda = (\lambda_1, 1)$.

2.2 Upper Bound. The generic trace ring for $M_{k,\ell}$ is a fixed ring for the general linear Lie superalgebra $pl(k, \ell)$. As such, it is contained in a fixed ring for $gl(k) \times gl(\ell) \subset pl(k, \ell)$. These facts were used in [1] to get an upper bound for $m^{\mathrm{mtr}}(M_{k,\ell})$, see corollary 21 of [1]. That corollary is stated in the language of contour integrals, but easily translates to the inner products of Schur functions, see section 1.2. Here is that translation in the case of $M_{2,1}$. Note that HS_λ denotes the hook Schur function defined in [2].

Lemma 9: *$m_\lambda(M_{2,1}) \leq m_\lambda^{\mathrm{mtr}}(M_{2,1})$ and*

$$m_\lambda^{\mathrm{mtr}}(M_{2,1}) \leq \Big\langle t(x_1, x_2, y) HS_\lambda\Big(1, 1, 1, \frac{x_1}{x_2}, \frac{x_2}{x_1}; \frac{x_1}{y}, \frac{x_2}{y}, \frac{y}{x_1}, \frac{y}{x_2}\Big), 1\Big\rangle_{2,1},$$

where $t(x_1, x_2, y) = (3 + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_1}{y} + \frac{x_2}{y} + \frac{y}{x_1} + \frac{y}{x_2})$.

By the definition of hook Schur functions

$$HS_\lambda(1, 1, 1, x_1 x_2^{-1}, x_1^{-1} x_2; x_1 y^{-1}, x_2 y^{-1}, x_1^{-1} y, x_2^{-1} y) =$$
$$\sum_{\mu \subset \lambda} S_\mu(1, 1, 1, x_1 x_2^{-1}, x_1^{-1} x_2) S_{(\lambda/\mu)'}(x_1 y^{-1}, x_2 y^{-1}, x_1^{-1} y, x_2^{-1} y). \qquad (10)$$

Also by Littlewood–Richardson,

$$S_{(\lambda/\mu)'}(x_1y^{-1},x_2y^{-1},x_1^{-1}y,x_2^{-1}y) = \sum_{\nu} C^{\lambda}_{\mu,\nu} S_{\nu'}(x_1y^{-1},x_2y^{-1},x_1^{-1}y,x_2^{-1}y). \tag{11}$$

The $C^{\lambda}_{\mu,\nu}$ are the Littlewood–Richardson coefficients. They equal the number of skew tableaux of shape λ/μ and content ν with certain properties. Since λ/μ has $|\nu|$ boxes

$$C^{\lambda}_{\mu,\nu} \leq |\nu|^{|\nu|}$$

and, in particular, if $|\nu|$ is bounded, so is $C^{\lambda}_{\mu,\nu}$. In equation (11) ν' must have height at most four since $S_{\nu'}(x_1y^{-1},\ldots,x_2^{-1}y)$ involves four variables. Hence, each row of ν is of length at most 4. On the other hand, $C^{\lambda}_{\mu,\nu}$ will be zero unless $\nu \subseteq \lambda$, so under the assumption that λ has height at most 5, ν must also have height at most 5. Hence, $\nu \subseteq (4^5)$, and in particular $|\nu|$ is bounded, and so $C^{\lambda}_{\mu,\nu}$ will also be bounded.

Now define the distance between two partitions α and β as $d(\alpha,\beta) = \sum |\alpha_i - \beta_i|$ The previous discussion implies that if λ is a partition of height at most 5, then

$$\begin{aligned} m_{\lambda}(M_{2,1}) &\leq m^{\text{mtr}}_{\lambda}(M_{2,1}) \\ &\leq \sum_{\mu,\nu} c_{\nu} \langle t(x_1,x_2,y) S_{\mu}(1,1,1,x_1x_2^{-1},x_1^{-1}x_2) \\ &\qquad \times S_{\nu}(x_1y^{-1},x_2y^{-1},x_1^{-1}y,x_2^{-1}y),1\rangle_{2,1}, \end{aligned} \tag{12}$$

where $d(\mu,\lambda) = O(1),\ |\nu| = O(1),\ c_{\nu} = O(1)$. To deal with the factor of t, note that

$$t(x_1,x_2,y) = S_{(1)}(1,1,1,x_1x_2^{-1},x_1^{-1}x_2) + S_{(1)}(x_1y^{-1},x_2y^{-1},x_1^{-1}y,x_2^{-1}y).$$

Substituting this into (12) gives a sum of two inner products. In the first expression, μ is replaced by partitions received from μ by adding one box, and in the Second ν is replaced by partitions received from ν by adding one box. The net effect is that the factor of t has no real effect on the big-O behaviour.

$$\begin{aligned} m_{\lambda}(M_{2,1}) &\leq m^{\text{mtr}}_{\lambda}(M_{2,1}) \\ &\leq \sum_{\mu,\nu} c_{\nu} \langle S_{\mu}(1,1,1,x_1x_2^{-1},x_1^{-1}x_2) S_{\nu}(x_1y^{-1},x_2y^{-1},x_1^{-1}y,x_2^{-1}y),1\rangle_{2,1}, \end{aligned} \tag{13}$$

where $d(\mu,\lambda) = O(1),\ |\nu| = O(1),\ c_{\mu} = O(1)$.

LEMMA 10: *For each partition μ, the inner product*

$$\langle S_\mu(x_1y^{-1}, x_2y^{-1}, x_1^{-1}y, x_2^{-1}y), 1\rangle_1,$$

can be written as a linear combination of the form

$$\sum_\nu c_i S_{(i)}(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2),$$

where $i \le \frac{1}{2}|\mu|$, and $|c_i|$ is bounded by a function of $|\mu|$.

Proof: For any Laurent polynomial in y, the GL(1) inner product with 1 is given by

$$\left\langle \sum a_n y^n, 1 \right\rangle = a_0.$$

Hence, for each $(x_1/y)^a(x_2/y)^b(y/x_1)^c(y/x_2)^d$ the inner product with 1 is zero unless $a+b=c+d$, in which case it is $x_1^{a-c}x_2^{b-d}$. Since $a-c=-(b-d)$ the inner product will be of total degree zero. In our case it will also be symmetric in x_1 and x_2. So it will a symmetric function in x_1/x_2 and x_2/x_1 and can be expressed as linear combination of functions $S_{(j)}(x_1/x_2, x_2/x_1)$ with $j \le a-c \le \frac{1}{2}|\mu|$.

To get the additional 1's, note the general fact that

$$S_a(x_1,\ldots,x_n) = S_a(x_1,\ldots,x_n,1) - S_{a-1}(x_1,\ldots,x_n,1).$$

Using this fact three times, the inner product $\langle S_\mu(x_1y^{-1}, x_2y^{-1}, x_1^{-1}y, x_2^{-1}y), 1\rangle_1$ is expressed as a linear combination of terms $S_\nu(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2)$. ∎

Substituting this lemma with equation (13) yields a linear combination of terms of the form

$$S_{(i)}(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2)S_\mu(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2).$$

We remark that, by Young's rule, such a product is a sum

$$\sum_\nu S_\nu(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2),$$

where $\nu \subseteq \mu$ and $|\mu/\nu| = i$. This implies our next lemma.

LEMMA 11: *If λ is a partition of height at most 5, then*

$$\begin{aligned} &m_\lambda(M_{2,1}) \\ &\quad \le \langle t(x_1,x_2,y) H S_\lambda(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2; x_1y^{-1}, x_2y^{-1}, x_1^{-1}y, x_2^{-1}y), 1\rangle_{2,1} \\ &\quad = \sum_\nu C_\nu \langle S_\nu(1,1,1,x_1x_2^{-1}, x_1^{-1}x_2), 1\rangle_2, \end{aligned}$$

where again $d(\lambda,\nu) = O(1)$ and $C_\nu = O(1)$.

Lemma 12: *The character sequence*

$$\sum_\lambda \sum_\nu C_\nu \langle S_\nu(1,1,1,x_1x_2^{-1},x_1^{-1}x_2),1\rangle_2 \chi^\lambda,$$

where C_ν is as the previous lemma, is the Young derived sequence of

$$\sum_\lambda \left\langle \sum_\nu C_\nu S_\nu(1,1,x_1x_2^{-1},x_1^{-1}x_2),1\right\rangle_2 \chi^\lambda.$$

Proof: $S_\nu(1,1,1,x_1x_2^{-1},x_1^{-1}x_2) = \sum_\xi S_\xi(1,1,x_1x_2^{-1},x_1^{-1}x_2)$ summed over all $\xi \subseteq \nu$ such that ν/ξ is a horizontal strip. ∎

Theorem 13: *If λ is a partition of height at most 5, $m_\lambda(M_{2,1})$ is the multiplicity of χ^λ in the cocharacter sequence of $M_{2,1}$ and $\overline{m}_\lambda(M_2)$ is the multiplicity of χ^λ in the mixed trace cocharacter sequence of 2×2 matrices, then $m_\lambda(M_{2,1}) \le C\mathcal{Y}(\overline{m}_\lambda(M_2))$, for some constant C.*

Proof: By the previous two lemmas, $m_\lambda(M_{2,1})$ is less than or equal to the Young derived sequence of

$$\sum_\nu C_\nu \langle S_\nu(1,1,x_1x_2^{-1},x_1^{-1}x_2),1\rangle_2.$$

By equation (4), this equals $\sum_\nu C_\nu m^{\mathrm{ptr}}(M_{2,1})$, which in turn is less than or equal to $\sum_\nu C_\nu m^{\mathrm{mtr}}(M_{2,1})$. To complete the proof we now need to show that

$$\sum_\nu C_\nu \bar{m}_\nu \le C\bar{m}_\lambda.$$

Since the number of partitions ν in the sum is uniformly bounded, as are the coefficients C_ν, we need only show that $\bar{m}_\nu$ is bounded by a constant times $\bar{m}_\lambda$. Let j be an upper bound on $d(\mu,\lambda)$ and recall that $\bar{m}_\lambda = \omega_1(\lambda)\omega_2(\lambda)\omega_3(\lambda)$ where each $\omega_i(\lambda) = \lambda_i - \lambda_{i+1} + 1$. It follows that $\omega_i(\mu) - \omega_i(\lambda) \le 2j$. Hence

$$\frac{\omega_i(\mu)}{\omega_i(\lambda)} = 1 + \frac{\omega_i(\mu) - \omega_i(\lambda)}{\omega_i(\lambda)} \le 1 + 2j$$

and so $\bar{m}_\mu \le (1+2j)^3 \bar{m}_\lambda$ and this completes the proof. ∎

Remark 14: If we replace the requirement that λ be a partition of height at most 5 with height of λ bounded by some fixed h the same proof shows

$$m_\lambda(M_{2,1}) \le C\bar{m}_\nu,$$

where C is a constant depending only on h, and $\nu = (\lambda_1,\ldots,\lambda_5)$ is the first 5 parts of λ.

References

[1] A. Berele, *Colength sequences for Kemer Algebras,* Journal of Algebra **293** (2005), 155–166.

[2] A. Berele and A. Regev, *Hook Young diagrams withapplications to combinatorics and to representations of Lie superalgebrs,* Advances in Mathematics **64** (1987), 118–175.

[3] A. Berele and A. Regev, *Codimensions of products and of intersections of verbally prime T-ideals,* Israel Journal of Mathematics **103** (1998), 17–28.

[4] V. Drensky, *Codimensions ot T-ideals and Hilbert series of relatively free algebras,* Journal of Algebra **91** (1984), 1–17.

[5] E. Formanek, *Invariants and the ring of generic matrices,* Journal of Algebra **89** (1984), 93–114.

[6] C. Procesi, *Computing with 2×2 matrices,* Journal of Algebra **87** (1984), 342–359.

[7] A. Regev, *Young derived sequences of S_n-characters,* Advances in Mathematics **106** (1994), 169–197.